The quantum Ashkin - Teller model with quantum group symmetry

1998 J. Phys. A: Math. Gen. 31 L495

(http://iopscience.iop.org/0305-4470/31/27/001)

J. Phys. A: Math. Gen. **31** (1998) L495–L499. Printed in the UK PII: S0305-4470(98)92335-6

LETTER TO THE EDITOR

The quantum Ashkin–Teller model with quantum group symmetry

Zhe Chang†‡‖, Han-Ying Guo†§ and Ke Wu†§

† CCAST (World Laboratory), PO Box 8730, 100080 Beijing, People's Republic of China
‡ Institute of High Energy Physics, Academia Sinica, PO Box 918(4), 100039 Beijing, People's Republic of China¶
§ Institute of Theoretical Physics, Academia Sinica, PO Box 2735, 100080 Beijing, People's Republic of China

Received 9 March 1998

Abstract. Relationships between quantum group and quantum universal enveloping algebra are investigated. We present what is called the quantum Ashkin–Teller model with general nearest-neighbour four interaction terms. In the case of vanishing four interaction, it reduces to two decoupled XXZ chains with surface terms, which has been studied thoroughly in the framework of quantum universal enveloping algebra symmetry. It is shown that the symmetry structure of the quantum version of Ashkin–Teller model is the quantum group $SL_q(2)$. This quantum group structure guarantees the integrability of the quantum model.

By using the integrability condition and writing the transfer matrix T of lattice models explicitly as $\binom{a\ b}{c\ d}$, Faddeev, Reshetikhin and Takhatajan (FRT) introduced a quantum group structure naturally [1]. In the FRT formalism, the algebraic relations of quantum group $SL_q(2)$ are of the form

$$\begin{aligned} &ab = qba \qquad ac = qca \qquad bd = qdb \\ &cd = qdc \qquad bc = cb \qquad ad - da = \lambda bc \\ &ad - qbc = 1 \qquad \lambda = q - q^{-1}. \end{aligned} \tag{1}$$

These equations can be rewritten into a matrix form,

$$R_{12}T_1T_2 = T_2T_1R_{12} \qquad \det_q T = 1 \tag{2}$$

where

$$R_{12} = \begin{pmatrix} q^{1/2} & & & \\ & q^{-1/2} & & \\ & q^{-1/2}\lambda & q^{-1/2} & \\ & & & q^{1/2} \end{pmatrix} \qquad T_1 = T \otimes 1 \qquad T_2 = 1 \otimes T.$$

‖ E-mail address: changz@bepc3.ihep.ac.cn
¶ Mailing address.

0305-4470/98/270495+05$19.50

The corresponding Hopf algebra structure of the quantum group $SL_q(2)$ is defined as

$$\begin{aligned}
&\Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a\otimes a + b\otimes c & a\otimes b + b\otimes d \\ c\otimes a + d\otimes c & c\otimes b + d\otimes d \end{pmatrix} \\
&\epsilon\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&s\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix}.
\end{aligned} \tag{3}$$

On the other hand, as a systematic method for solving the Yang–Baxter equation [2, 3], the quantum universal enveloping algebra [4, 5] has been intensively studied. The quantum universal enveloping algebra $U_q(sl(2))$ is a Hopf algebra which is neither commutative nor cocommutative with generators k, k^{-1}, e, f and 1. The corresponding algebraic relations are

$$\begin{aligned}
&ke = qek \qquad kf = q^{-1}fk \qquad k^{-1}k = 1 \qquad [e, f] = \frac{k^2 - k^{-2}}{q - q^{-1}} \\
&\Delta(k) = k\otimes k \qquad \Delta(e) = e\otimes k^{-1} + k\otimes e \qquad \Delta(f) = f\otimes k^{-1} + k\otimes f \\
&\epsilon(k) = 1 \qquad \epsilon(e) = \epsilon(f) = 0 \\
&s(k) = k^{-1} \qquad s(e) = -q^{-1}e \qquad s(f) = -qf.
\end{aligned} \tag{4}$$

It was found that the quantum universal enveloping algebra has many applications in physics systems [6]. For example, in a Hamiltonian system it is an enlarged symmetry that maintains invariance of equations of motion and allows a deformation of the Hamiltonian and symplictic form; the configuration space of the integrable lattice model can be analysed in terms of the representation theory of the quantum universal enveloping algebra. The quantum symmetry approach based on the quantum universal enveloping algebra is now a popular topic in different fields of modern physics.

It may be some surprise to notice that there are few physical applications of the quantum group, which is a more physical origin than the quantum universal enveloping algebra. In FRT formalism, the quantum group can be identified as a symmetry structure of the transfer matrix for integrable lattice models. Thus, an important topic in the field is to discuss the relationships of quantum universal enveloping algebra and quantum group and then to investigate possible physical applications of the quantum group. Other reasons for investigating relationships between quantum group and quantum universal enveloping algebra include opening a way of constructing classical realization of quantum group because the realizations of the quantum universal enveloping algebra in classical physics systems have been set up [7–11].

In this letter, we present what is called the quantum Ashkin–Teller model with general nearest-neighbour four interacting terms, which is shown to possess quantum group $SL_q(2)$ symmetry. The quantum group is constructed from the tensor product of two sets of independent quantum universal enveloping algebras $U_q(sl(2))$. The quantum integrability is guaranteed by this quantum group structure.

The Hamiltonian of the quantum Ashkin–Teller [12, 13] model in the one-dimensional lattice is of the form

$$\begin{aligned}
H = {} & \frac{K_{2\sigma}}{\sin\eta}\sum_{i=1}^{N}(\sigma_i^1\sigma_{i+1}^1 + \sigma_i^2\sigma_{i+1}^2 + \cos\eta\,\sigma_i^3\sigma_{i+1}^3 + \mathrm{i}\sin\eta(\sigma_i^3 - \sigma_{i+1}^3) - \cos\eta) \\
& + \frac{K_{2\tau}}{\sin\eta}\sum_{i=1}^{N}(\tau_i^1\tau_{i+1}^1 + \tau_i^2\tau_{i+1}^2 + \cos\eta\,\tau_i^3\tau_{i+1}^3 + \mathrm{i}\sin\eta(\tau_i^3 - \tau_{i+1}^3) - \cos\eta)
\end{aligned}$$

$$+\frac{K_4}{\sin^2\eta}\sum_{i=1}^{N}(\sigma_i^1\sigma_{i+1}^1+\sigma_i^2\sigma_{i+1}^2+\cos\eta\,\sigma_i^3\sigma_{i+1}^3+\mathrm{i}\sin\eta(\sigma_i^3-\sigma_{i+1}^3)-\cos\eta)$$
$$\times(\tau_i^1\tau_{i+1}^1+\tau_i^2\tau_{i+1}^2+\cos\eta\,\tau_i^3\tau_{i+1}^3+\mathrm{i}\sin\eta(\tau_i^3-\tau_{i+1}^3)-\cos\eta) \tag{5}$$

where we have used the notations

$$\sigma_k^i=(1_{(1)_\sigma}\otimes 1_{(1)_\tau})\otimes(1_{(2)_\sigma}\otimes 1_{(2)_\tau})\otimes\cdots\otimes(\sigma_{(k)}^i\otimes 1_{(k)_\tau})\otimes\cdots\otimes(1_{(N)_\sigma}\otimes 1_{(N)_\tau})$$
$$\tau_k^i=(1_{(1)_\sigma}\otimes 1_{(1)_\tau})\otimes(1_{(2)_\sigma}\otimes 1_{(2)_\tau})\otimes\cdots\otimes(1_{(k)_\sigma}\otimes \tau_{(k)}^i)\otimes\cdots\otimes(1_{(N)_\sigma}\otimes 1_{(N)_\tau}).$$

Here σ^i, τ^i $(i=1,2,3)$ are two sets of independent Pauli matrices ($[\sigma^i,\tau^j]=0$) and η is a free parameter.

If the coupling constant $K_4=0$, the system reduces to two decoupled one-dimensional XXZ models with boundary terms. The XXZ chain with boundary terms has recently been studied thoroughly, and it was found that it possesses quantum universal enveloping algebra $U_q(sl(2))$ symmetry [14, 15]. This is to say that we have $[H_{XXZ},U_q(sl(2))]=0$. The generators of the quantum universal enveloping algebra in $(V_\sigma^{1/2}\otimes V_\tau^{1/2})^{\otimes N}$ is defined as

$$S_\sigma^\pm=\sum_{i=1}^{N}(q^{s^3_{(1)\sigma}}\otimes 1_{(1)_\tau})\otimes\cdots\otimes(q^{s^3_{(i-1)\sigma}}\otimes 1_{(i-1)_\tau})$$
$$\otimes(s^\pm_{(i)_\sigma}\otimes 1_{(i)_\tau})\otimes(q^{-s^3_{(i+1)\sigma}}\otimes 1_{(i+1)_\tau})\otimes\cdots\otimes(q^{-s^3_{(N)\sigma}}\otimes 1_{(N)_\tau})$$
$$q^{S^3_\sigma}=(q^{s^3_{(1)\sigma}}\otimes 1_{(1)_\tau})\otimes(q^{s^3_{(2)\sigma}}\otimes 1_{(2)_\tau})\otimes\cdots\otimes(q^{s^3_{(N)\sigma}}\otimes 1_{(N)_\tau})$$
$$s^i_\sigma=\tfrac{1}{2}\sigma^i$$

and

$$S_\tau^\pm=\sum_{i=1}^{N}(1_{(1)_\sigma}\otimes q^{s^3_{(1)\tau}})\otimes\cdots\otimes(1_{(i-1)_\sigma}\otimes q^{s^3_{(i-1)\tau}})$$
$$\otimes(1_{(i)_\sigma}\otimes s^\pm_{(i)_\tau})\otimes(1_{(i+1)_\sigma}\otimes q^{-s^3_{(i+1)\tau}})\otimes\cdots\otimes(1_{(N)_\sigma}\otimes q^{-s^3_{(N)\tau}})$$
$$q^{S^3_\tau}=(1_{(1)_\sigma}\otimes q^{s^3_{(1)\tau}})\otimes(1_{(2)_\sigma}\otimes q^{s^3_{(2)\tau}})\otimes\cdots\otimes(1_{(N)_\sigma}\otimes q^{s^3_{(N)\tau}})$$
$$s^i_\tau=\tfrac{1}{2}\tau^i.$$

It is easy to check that the generators $S^\pm_\sigma$ and S^3_σ ($S^\pm_\tau$ and S^3_τ) satisfy the quantum universal enveloping algebra relations of $U_q^{\sigma(\tau)}(sl(2))$ as well as associated Hopf algebra structure. By using the $R^{\sigma(\tau)}$ matrix

$$R_{12}^{\sigma(\tau)}=\begin{pmatrix} q^{1/2} & & & \\ & q^{-1/2} & & \\ & q^{-1/2}\lambda & q^{-1/2} & \\ & & & q^{1/2}\end{pmatrix} \tag{6}$$

we can write the quantum universal enveloping algebra $U_q^{\sigma(\tau)}(sl(2))$ into a more abstract form [16, 17]

$$R_{12}^{\sigma(\tau)}L^\pm_{\sigma(\tau)_1}L^\pm_{\sigma(\tau)_2}=L_2^\pm L_1^\pm R_{12}\qquad R_{12}^{\sigma(\tau)}L^-_{\sigma(\tau)_1}L^+_{\sigma(\tau)_2}=L^+_{\sigma(\tau)_2}L^-_{\sigma(\tau)_1}R_{12}^{\sigma(\tau)}$$
$$\Delta_{\sigma(\tau)}(L^\pm_{\sigma(\tau)})=L^\pm_{\sigma(\tau)}\dot{\otimes}L^\pm_{\sigma(\tau)}$$
$$s_{\sigma(\tau)}(L^+_{\sigma(\tau)})=\begin{pmatrix} q^{S^3_{\sigma(\tau)}} & -q\lambda S^+_{\sigma(\tau)} \\ & q^{-S^3_{\sigma(\tau)}}\end{pmatrix}\qquad s_{\sigma(\tau)}(L^-_{\sigma(\tau)})=\begin{pmatrix} q^{-S^3_{\sigma(\tau)}} & \\ -q\lambda S^+_{\sigma(\tau)} & q^{S^3_{\sigma(\tau)}}\end{pmatrix} \tag{7}$$
$$\epsilon_{\sigma(\tau)}(L^\pm_{\sigma(\tau)})=1$$

where we have used the notations

$$L^+_{\sigma(\tau)} = \begin{pmatrix} q^{-S^3_{\sigma(\tau)}} & \lambda S^+_{\sigma(\tau)} \\ & q^{S^3_{\sigma(\tau)}} \end{pmatrix} \qquad L^-_{\sigma(\tau)} = \begin{pmatrix} q^{S^3_{\sigma(\tau)}} & \\ -\lambda S^-_{\sigma(\tau)} & q^{-S^3_{\sigma(\tau)}} \end{pmatrix}$$

and the operation $\dot{\otimes}$ between two matrices A and B is defined as

$$(A\dot{\otimes}B)_{ij} = A_{ik} \otimes B_{kj}.$$

By using the operators $L^\pm_{\sigma(\tau)}$, which are defined on the space $V^{(1/2)}_\sigma$ and $V^{(1/2)}_\tau$, respectively, we can introduce four (and only four) nontrivial tensor operators on the space $V^{(1/2)}_\sigma \otimes V^{(1/2)}_\tau$, i.e. $L^+_\sigma L^+_\tau$, $L^+_\sigma L^-_\tau$, $L^-_\sigma L^+_\tau$, and $L^-_\sigma L^-_\tau$. In the following (to agree with FRT's notation), we denote these tensor operators as T and write them explicitly as

$$T = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}. \tag{8}$$

From equation (7), we know that the new operators T satisfy the equation

$$R_{12}T_1T_2 = T_2T_1R_{12} \qquad \det_q T = 1. \tag{9}$$

This can be verified straightforwardly. For example, for $T = L^+_\sigma \dot{\otimes} L^-_\tau$, we have

$$\begin{aligned} a'b' &= (l^+_{\sigma(11)} \otimes l^-_{\tau(11)} + l^+_{\sigma(12)} \otimes l^-_{\tau(21)}) \cdot (l^+_{\sigma(12)} \otimes l^-_{\tau(22)}) \\ &= l^+_{\sigma(11)} l^+_{\sigma(12)} \otimes l^-_{\tau(11)} l^-_{\tau(22)} + l^+_{\sigma(12)} l^+_{\sigma(12)} \otimes l^-_{\tau(21)} l^-_{\tau(22)} \\ &= q l^+_{\sigma(12)} l^+_{\sigma(11)} \otimes l^-_{\tau(22)} l^-_{\tau(11)} + q l^+_{\sigma(12)} l^+_{\sigma(12)} \otimes l^-_{\tau(22)} l^-_{\tau(21)} \\ &= q(l^+_{\sigma(12)} \otimes l^-_{\tau(22)}) \cdot (l^+_{\sigma(11)} \otimes l^-_{\tau(11)} + l^+_{\sigma(12)} \otimes l^-_{\tau(21)}) \\ &= q b'a' \end{aligned}$$

and

$$\begin{aligned} a'd' - qb'c' = {} & (l^+_{\sigma(11)} \otimes l^-_{\tau(11)} + l^+_{\sigma(12)} \otimes l^-_{\tau(21)}) \cdot (l^+_{\sigma(22)} \otimes l^-_{\tau(22)}) \\ & - q(l^+_{\sigma(12)} \otimes l^-_{\tau(22)}) \cdot (l^+_{\sigma(22)} \otimes l^-_{\tau(21)}) \\ = {} & l^+_{\sigma(11)} l^+_{\sigma(22)} \otimes l^-_{\tau(11)} l^-_{\tau(22)} + l^+_{\sigma(12)} l^+_{\sigma(22)} \otimes l^-_{\tau(21)} l^-_{\tau(22)} - q l^+_{\sigma(12)} l^+_{\sigma(22)} \otimes l^-_{\tau(22)} l^-_{\tau(21)} \\ = {} & 1 \end{aligned}$$

and so on.

Let $P^\otimes$ be the transposition operator [18] in $V^{(1/2)} \otimes V^{(1/2)}$, which satisfies

$$P^\otimes : A\dot{\otimes}B \longrightarrow B\dot{\otimes}A. \tag{10}$$

Then the comultiplication Δ for these tensor operators T can be introduced as

$$\Delta = (\mathrm{id} \otimes P^\otimes \otimes \mathrm{id})(\Delta_\sigma \dot{\otimes} \Delta_\tau). \tag{11}$$

As an example, for $T = L^+_\sigma \dot{\otimes} L^-_\tau$, it is easy to check that

$$\begin{aligned} \Delta \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} &= (\mathrm{id} \otimes P^\otimes \otimes \mathrm{id})(\Delta_{L^+_\sigma} \dot{\otimes} \Delta_{L^-_\tau})(L^+_\sigma \dot{\otimes} L^-_\tau) \\ &= (\mathrm{id} \otimes P^\otimes \otimes \mathrm{id})(L^+_\sigma \dot{\otimes} L^+_\sigma \dot{\otimes} L^-_\tau \dot{\otimes} L^-_\tau) \\ &= (L^+_\sigma \dot{\otimes} L^-_\sigma) \dot{\otimes} (L^+_\tau \dot{\otimes} L^-_\tau) \\ &= \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \dot{\otimes} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}. \end{aligned}$$

Almost the same relations for other three tensor operators T can also be obtained in the same way. Therefore, in general, we have

$$\Delta T = T \dot{\otimes} T. \tag{12}$$

Define the co-unit operator ϵ as

$$\epsilon = \epsilon_\sigma \otimes \epsilon_\tau. \tag{13}$$

Then, we have

$$\epsilon T = 1. \tag{14}$$

Finally, the antipode operator s is of the form,

$$s = P(s_\tau \otimes s_\sigma)P^{\otimes}. \tag{15}$$

A straightforward calculation gives that

$$s\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} d' & -q^{-1}b' \\ -qc' & a' \end{pmatrix}. \tag{16}$$

Now, we are in a position to conclude that the symmetry of the quantum Ashkin–Teller-like model is the quantum group $SL_q(2)$, i.e. $[H, T] = 0$. It should be noted that there are four types of the tensor operators T and all of them satisfy the quantum group relations strictly.

It is well known that the decoupled system can be solved by using the quantum universal enveloping algebra approach [19]. Now, we know that the only symmetry structure of the quantum Ashkin–Teller-like model is the quantum group $SL_q(2)$. There are infinite physical quantities which commute with the Hamiltonian and thus this system is integrable. A group analogy of the quantum universal enveloping algebra symmetry approach to the system is apparent. Further investigations in this direction are in progress [20].

The authors would like to thank G X Ju, X C Song and S K Wang for useful discussions. The work was supported by the National Science Foundation of China.

References

[1] Faddeev L D, Reshetikhin N Yu and Takhtajan L A 1990 *Leningrad Math. J.* **1** 193
[2] Yang C N 1967 *Phys. Rev. Lett.* **19** 1312
[3] Baxter R J 1972 *Ann. Phys., NY* **70** 193
[4] Drinfel'd V G 1987 Quantum groups *Proc. ICM (Berkeley, CA)* (Providence, RI: American Mathematical Society) p 798
[5] Jimbo M 1985 *Lett. Math. Phys.* **10** 63
[6] For the review, see Chang Z 1995 *Phys. Rep.* **262** 137
[7] Chang Z, Chen W and Guo H Y 1990 *J. Phys. A: Math. Gen.* **23** 4185
[8] Chang Z, Chen W, Guo H Y and Yan H 1990 *J. Phys. A: Math. Gen.* **23** 5371
[9] Chang Z, Chen W, Guo H Y and Yan H 1991 *J. Phys. A: Math. Gen.* **24** 1427
[10] Chang Z, Chen W, Guo H Y and Yan H 1991 *J. Phys. A: Math. Gen.* **24** 5435
[11] Chang Z, Wang J X and Yan H 1991 *J. Math. Phys.* **32** 3241
[12] Ashkin J and Teller E 1943 *Phys. Rev.* **64** 178
[13] Kohmoto M, den Nijs M and Kadanoff L P 1981 *Phys. Rev.* B **24** 5229
[14] Alcaraz F C, Barber M N, Batchelor M T, Baxter R J and Quispel G R W 1987 *J. Phys. A: Math. Gen.* **20** 6397
[15] Pasquier V and Saleur H 1990 *Nucl. Phys.* B **330** 523
[16] Wu K, Guo H Y and Zhang R J 1993 *High Energy Nucl. Phys.* **17** 262 (in Chinese)
[17] Wu K and Guo H Y 1993 *High Energy Nucl. Phys.* **17** 796 (in Chinese)
[18] Podles P and Woronowicz S L 1990 *Commun. Math. Phys.* **130** 381
[19] Hou B Y, Hou B Y and Ma Z Q 1991 *J. Phys. A: Math. Gen.* **24** 2847
[20] Chang Z, Guo H Y and Wu K Quantum groups via quantum algebras, in preparation